*International Journal of Theoretical Physics, Vol. 27, No. 1, 1988*

# Mass Spectral Condition for Leptons and Quarks

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*Received August 27, 1987*

Consideration is given to the mass spectral condition $a(a_{,m})a^{\dagger} = a_{,m}$, where $a$ is the annihilation operator for $Q = -1$ states in the Hilbert space of leptons and quarks and $a_{,m}$ is the commutator resolvent of $a$ with respect to $m$. It is observed that this spectral condition, which simply requires $a$ to be a congruent automorph of $a_{,m}$, implies that the third-, fourth-, and fifth-generation $Q = -1$ leptons have the masses 1788.03 MeV, 42.1649 GeV, and 1.33422 TeV, respectively. With the assumption that the mass spectral condition also holds for $Q = 0$ states in the Hilbert space, one obtains new theoretical upper and lower bounds on the neutrino masses.

## 1. INTRODUCTION

In recent years Fleming (1981), Pfeifer (1981), Jensen (1984), and Eidus (1985) have advanced applied Hilbert space spectral theory for quantum mechanical systems by deriving associated operator conditions that restrict or fix energy spectra. Since electroweak effects in Bhabha scattering ($e^+e^- \to e^+e^-$) at a center-of-mass energy of 29 GeV rule out substructure for the electron up to mass scales of 1 TeV (see, for example, Fernandez *et al.*, 1987), it is reasonable to seek an associated operator condition that fixes the $Q = -1$ sequential lepton mass spectra.

The present paper shows that one can indeed formulate an abstract mass spectral condition in a Hilbert space for lepton and quark states. Expressed below by equation (8), this spectral condition is shown to be consistent with the $\tau$ mass value and the nonexistence of the fourth-generation charged lepton up to 41 GeV, in addition to being consistent with estimated quark mass values and all bounds on the neutrino masses. Presumably the spectral condition (8) must arise in a nonlinear quantum field theory for the fundamental fermions, and the experimental detection of the fourth-generation charged lepton at 42.165 GeV, as predicted by (8),

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0020-7748/88/0100-0027$06.00/0 

would clearly motivate the development of a nonlinear quantum field theory that engenders (8). However, the logically simple abstract approach taken here does not invoke model-dependent quantum field theory assumptions, but defers physical completeness in favor of mathematical exactness.

## 2. FORMULATION AND IMPLICATIONS OF THE MASS SPECTRAL CONDITION

According to the standard model, the leptons and quarks can be arranged in a periodic table with each generation appearing as a row having eight entries (see Table I). In analogy to the atomic number for the elements, one can assign a principal quantum number $n$ to each lepton and quark, as shown in Table I, where $n$ increases with increasing charge number $Q$ in each successive generation. Thus, the $Q=-1$ sequential leptons are assigned principal quantum numbers $n=8k=0, 8, 16, 24, \ldots$.

Consider a Hilbert space $\mathscr{H}$ in which the leptons and quarks are represented by state vectors. Let the subspace of $\mathscr{H}$ for the lepton or quark with principal quantum number $n$ be denoted by $|n\rangle$, a normalized eigenket of the self-adjoint mass operator $m \equiv m^{\dagger}$:

$$m|n\rangle = m_n|n\rangle, \qquad \langle n|n'\rangle = \delta_{nn'} \tag{1}$$

In particular, the subspaces for the $Q=-1$ leptons are

$$e\colon |0\rangle, \qquad \mu\colon |8\rangle, \qquad \tau\colon |16\rangle, \qquad \delta\colon |24\rangle, \ldots \tag{2}$$

with the experimental mass values (Aguilar-Benitez *et al.*, 1986; Cline and Mohammadi, 1987; M. Perl, private communication, 1987)

$$m_0 = 0.511 \text{ MeV}, \qquad m_8 = 105.659 \text{ MeV},$$
$$m_{16} = 1784.2\ (\pm 3.2) \text{ MeV}, \qquad m_{24} > 41 \text{ GeV}. \tag{3}$$

**Table I.** First Four Rows (Generations) of the Periodic Table for Fundamental Fermions[a].

| $Q$ | $-1$ | $-1/3$ | | | $0$ | $+2/3$ | | |
|---|---|---|---|---|---|---|---|---|
| First generation | 0<br>$e$ | 1<br>$d'_1$ | 2<br>$d'_2$ | 3<br>$d'_3$ | 4<br>$\nu_e$ | 5<br>$u_1$ | 6<br>$u_2$ | 7<br>$u_3$ |
| Second generation | 8<br>$\mu$ | 9<br>$s'_1$ | 10<br>$s'_2$ | 11<br>$s'_3$ | 12<br>$\nu_\mu$ | 13<br>$c_1$ | 14<br>$c_2$ | 15<br>$c_3$ |
| Third generation | 16<br>$\tau$ | 17<br>$b'_1$ | 18<br>$b'_2$ | 19<br>$b'_3$ | 20<br>$\nu_\tau$ | 21<br>$t_1$ | 22<br>$t_2$ | 23<br>$t_3$ |
| Fourth generation | 24<br>$\delta$ | 25<br>$h'_1$ | 26<br>$h'_2$ | 27<br>$h'_3$ | 28<br>$\nu_\delta$ | 29<br>$g_1$ | 30<br>$g_2$ | 31<br>$g_3$ |

[a] Principal quantum number $n$ is assigned to the leptons and tricolored quarks as shown above each particle symbol. The $Q=-1$ leptons appear in the first column with $n=8k$, $k=0, 1, 2, \ldots$

In the $Q=-1$ sector of $\mathscr{H}$ spanned by the subspace kets in (2), the "annihilation" and "creation" operators are given, respectively, by

$$
\begin{aligned}
a &\equiv \sum_{k=0}^{\infty} (8k+8)^{1/2}|8k\rangle\langle 8k+8| \\
a^{\dagger} &= \sum_{k=0}^{\infty} (8k+8)^{1/2}|8k+8\rangle\langle 8k|
\end{aligned}
\tag{4}
$$

and the principal quantum number operator is expressible in the $Q=-1$ sector of $\mathscr{H}$ in terms of (4) as

$$n \equiv a^{\dagger}a = \sum_{k=1}^{\infty} 8k|8k\rangle\langle 8k| \tag{5}$$

Denoted as $a_{,m}$ in the following, the *commutator resolvent of a with respect to m* is defined implicitly by the equations

$$
\begin{aligned}
(a_{,m})m - m(a_{,m}) &= a \\
\langle n|a|n'\rangle = 0 &\Rightarrow \langle n|a_{,m}|n'\rangle = 0
\end{aligned}
\tag{6}
$$

and is obtained from (1) and the first member of (4) as

$$a_{,m} = \sum_{k=0}^{\infty} (8k+8)^{1/2}(m_{8k+8}-m_{8k})^{-1}|8k\rangle\langle 8k+8| \tag{7}$$

The operator $A$ is said to be a *congruent automorph* of the operator $B$ if $ABA^{\dagger}=B$. Thus, for example, any unitary operator is a congruent automorph of the identity operator in a Hilbert space. More generally, the operator $B$ is invariant in form with respect to the transformation of basis vectors of the space by $A$ if and only if $A$ is a congruent automorph of $B$. Nontrivial examples and applications of congruent automorphs are numerous in the classical and contemporary literature on inner-product vector spaces.

With these notations and definitions prescribed, consider:

*Mass Spectral Condition.* The $Q=-1$ annihilation operator $a$ [defined in (4)] is a congruent automorph of its commutator resolvent $a_{,m}$ [displayed in (7)]:

$$a(a_{,m})a^{\dagger} = a_{,m} \tag{8}$$

The mass spectral condition (8) simply states that $a_{,m}$ is invariant with respect to the "shear" of the $Q=-1$ sector of $\mathscr{H}$ produced by the action of $a$.

From (7) and the second member of (4) one obtains

$$(a_{,m})a^{\dagger} = \sum_{k=0}^{\infty} (8k+8)(m_{8k+8} - m_{8k})^{-1}|8k\rangle\langle 8k| \tag{9}$$

and hence the left-hand side of (8) can be expressed as an eigenket series by applying the first member of (4) to (9):

$$a(a_{,m})a^{\dagger} = \sum_{k=0}^{\infty} (8k+8)^{1/2}(8k+16)(m_{8k+16} - m_{8k+8})^{-1}|8k\rangle\langle 8k+8| \tag{10}$$

By comparing the coefficients in (10) and (7), it follows that the mass spectral condition (8) is satisfied if and only if

$$(8k+16)(m_{8k+16} - m_{8k+8})^{-1} = (m_{8k+8} - m_{8k})^{-1} \tag{11}$$

for all integer $k \geq 0$. The recurrence relation (11) immediately produces the mass separation equation

$$m_{8k+8} - m_{8k} = 8^k(k+1)!(m_8 - m_0) \tag{12}$$

and the associated mass formula

$$m_{8k} = m_0 + (m_8 - m_0) \sum_{l=1}^{k} 8^{l-1} l! \tag{13}$$

for all $k \geq 1$. By substituting the empirical values for $m_0$ and $m_8$ shown in (3), one obtains from formula (13) the masses for the heavier sequential leptons:

$$\begin{aligned} k=2: &\quad m_{16} = 0.511 + 105.148(17) = 1788.03 \text{ Mev} \\ k=3: &\quad m_{24} = 0.511 + 105.148(401) = 42.1649 \text{ GeV} \\ k=4: &\quad m_{32} = 0.511 + 105.148(12{,}689) = 1.33422 \text{ TeV} \end{aligned} \tag{14}$$

The theoretical mass $m_{16}$ shown in (14) is consistent with the experimental $\tau$ value in (3), while the fourth-generation $\delta$ mass $m_{24} = 42.1649$ GeV lies in the current search range of the UA1 collaboration at CERN and may be detected presently. Assuming that the mass spectral condition (8) is maintained up to a TeV mass scale, the next sequential lepton is predicted to have the mass $m_{32} = 1.33422$ TeV; for production and detection of this fifth-generation lepton, one would require a high-energy collider (such as the SSC) that exceeds the extended range of the Fermilab Tevatron.

It can be conjectured that the mass spectral condition (8) also applies to the quark and neutrino sector of $\mathscr{H}$ if the annihilation and creation operators (4) are redefined for the sector of interest by making the replacement $8k \to (8k+j)$ with $j = 1, \ldots, 7$. The generalization of (13) then follows directly as

$$m_{8k+j}=m_j+(m_{8+j}-m_j)\left[\sum_{l=1}^{k}8^{l-1}\Gamma\left(l+1+\frac{1}{8}j\right)\Big/\Gamma\left(2+\frac{1}{8}j\right)\right] \quad (15)$$

where $\Gamma(x)\,[=(x-1)!$ for $x$ a positive integer] is the standard gamma function. For the quark sectors with $j=1,2,3$ and $5,6,7$ (see Table I), it is easily verified that (15) is qualitatively consistent with QCD estimated values for the quark masses, but an assessment of the accuracy of theoretical agreement is impeded by the well-known uncertainties associated with the quark mass values. On the other hand, the $j=4$, $Q=0$ case (see Table I) admits two significant theoretical bounds on neutrino mass, which derive directly from (15) by the following considerations.

First, the current experimental upper bounds on the electron and muon neutrino masses (Aguilar-Benitez *et al.*, 1986)

$$m_4<46\text{ eV},\qquad m_{12}<0.25\text{ MeV} \quad (16)$$

imply that

$$m_{8k+4}<46\text{ eV}+(0.25\text{ MeV})\left[\sum_{l=1}^{k}8^{l-1}\Gamma\left(l+\frac{3}{2}\right)\Big/\Gamma\left(\frac{5}{2}\right)\right] \quad (17)$$

as a consequence of (15) with $j=4$. Hence, by setting $k=2$ one finds the theoretical upper bound on the tau neutrino mass

$$m_{20}<5.25\text{ MeV} \quad (18)$$

which is more than an order of magnitude sharper that the current empirical upper bound $m_{20}<70$ MeV. Second, the decay width of the gauge boson $Z$ in the standard model indicates the existence of fewer than seven "light" neutrinos with masses less than 10 GeV, or equivalently

$$m_{52}\equiv(m_{8k+4})_{k=6}>10\text{ GeV} \quad (19)$$

holds as a conservative lower bound on the seventh-generation neutrino mass. Therefore, by combining (19) and (15) for $j=4$ and $k=6$, one obtains a theoretical *lower* bound on the $\nu_\mu-\nu_e$ mass separation

$$(m_{12}-m_4)>(10\text{ GeV})\left[\sum_{l=1}^{6}8^{l-1}\Gamma\left(l+\frac{3}{2}\right)\Big/\Gamma\left(\frac{5}{2}\right)\right]^{-1}\simeq 217\text{ eV} \quad (20)$$

and concomitant lower bounds (e.g., $m_{20}>4.56$ keV) on the neutrinos with $k\gtrsim 2$. The bounds displayed in (18) and (20) may be subject to experimental tests in the near future.

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